LIMIT PROBABILITIES IN A MULTI-TYPE CRITICAL AGE-DEPENDENT BRANCHING PROCESS

BY

HOWARD J. WEINER

TECHNICAL REPORT NO. 2
MARCH 7, 1977

PREPARED UNDER GRANT
DAAG29-77-G-0031
FOR THE U.S. ARMY RESEARCH OFFICE

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

LIMIT PROBABILITIES IN A MULTI-TYPE
CRITICAL AGE-DEPENDENT BRANCHING PROCESS

by

HOWARD J. WEINER

TECHNICAL REPORT NO. 2
MARCH 7, 1977

Prepared under Grant DAAG29-77-G-0031
For the U. S. Army Research Office
Herbert Solomon, Project Director

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

Partially supported under Office of Naval Research
Contract N00014-76-C-0475 (NR-042-267) and issued as Technical Report No. 242.

Limit Probabilities in a Multi-type Critical Age-Dependent Branching Process

by

Howard J. Weiner

I. Introduction and Assumptions

Let

(1.1) $Z_{ij}(t)$ = the number of cells of type j at time t starting with one new-born cell of type i at $t = 0$ with $1 \leq i \leq m$ in an m-type critical age-dependent branching process described as follows.
At time $t = 0$, one newly born cell of type i starts the process, for some $1 \leq i \leq m$. The cell lives a random time described by a continuous distribution function

$$G_i(t),\ G_i(0+) = 0. \tag{1.2}$$

At the end of its life, cell i is replaced by j_1 new daughter cells of type 1, j_2 new cells of type $2, \ldots, j_m$ cells of type m with probability $p_{ij_1j_2j_3\cdots j_m}$.

Define the generating functions, for $\underline{s} = (s_1,\ldots,s_m)$, $\underline{j} = (j_1,\ldots,j_m)$, $\underline{s}^{\underline{j}} \equiv (s_1^{j_1},\ldots,s_2^{j_m})$,

$$h_i(s_1,\ldots s_m) \equiv h_i(\underline{s}) = \sum_{(j_1\cdots j_m)} p_{ij_1\cdots j_m} s_1^{j_1}\cdots s_m^{j_m} \equiv \sum_{\underline{j}} p_{i\underline{j}}\underline{s}^{\underline{j}} . \tag{1.3}$$

Each daughter cell proceeds independently of the state of the system, with each cell type j governed by $G_j(t)$ and $h_j(\underline{s})$.

Assume, for $\underline{1+\varepsilon} \equiv (1+\varepsilon,\ldots,1+\varepsilon)$ and $\underline{1} = (1,\ldots,1)$, m-vectors,

$$h_i(\underline{1+\varepsilon}) < \infty \quad \text{for} \quad 1 \leq i \leq m. \tag{1.4}$$

This insures that all moments of $h_i(\underline{s})$ evaluated at $\underline{s} = \underline{1}$ may be computed by partial differentiations under the summation sign.

Define, for $1 \leq i,\ j \leq m$,

$$m_{ij} \equiv \left.\frac{\partial h_i(\underline{s})}{\partial s_j}\right|_{\underline{s}=\underline{1}} \equiv h_{ij}(\underline{1}) \tag{1.5}$$

and assume

$$m_{ij} > 0 \quad \text{all} \quad 1 \leq i,\ j \leq m, \tag{1.6}$$

and let the first moment $m \times m$ matrix be

$$M = (m_{ij}). \tag{1.7}$$

By standard Frobenius theory [2], there is a largest eigenvalue in absolute value, denoted ρ, which is positive.

<u>The basic assumption of criticality is that</u>

$$\rho = 1. \tag{1.8}$$

It follows again from standard Frobenius theory [2] that there are strictly positive eigenvectors $\underline{u} > 0$, $\underline{v} > 0$ (no notational distinction between row and column m-vectors) such that

(1.9)
$$\underline{M}\underline{u} = \underline{u}$$
$$\underline{v}M = \underline{v}$$
$$\underline{u} \cdot \underline{1} \equiv \sum_{i=1}^{m} u_i = 1$$
$$\underline{u} \cdot \underline{v} \equiv \sum_{i=1}^{m} u_i v_i = 1.$$

Let the quantity denoted by $Q(\underline{u})$ satisfy

(1.10)
$$0 < Q(\underline{u}) \equiv \frac{1}{2} \sum_{i=1}^{m} \sum_{\ell=1}^{m} \sum_{r=1}^{m} \frac{\partial^2 h_i(\underline{1})}{\partial s_\ell \partial s_r} u_\ell u_r v_i < \infty$$

and also

(1.11)
$$\frac{\partial^2 h_i(\underline{1})}{\partial s_j \partial s_k} > 0 \quad 1 \leq i,\ j,k \leq m.$$

Further assume that each distribution function $G_i(t)$ satisfies

(1.12)
$$\int_0^\infty t^{4+\delta} dG_i(t) < \infty \quad \text{for some } \delta > 0.$$

Let

(1.13)
$$0 < \mu_i \equiv \int_0^\infty t\, dG_i(t) < \infty,$$

and denote

(1.14)
$$F_i(\underline{s},t) \equiv E\left[s_1^{Z_{i1}(t)} s_2^{Z_{i2}(t)} \cdots s_m^{Z_{im}(t)}\right].$$

Let

(1.15)
$$\underline{Z}_i(t) = (Z_{i1}(t), Z_{i2}(t), \ldots, Z_{im}(t))$$

Let

(1.16)
$$\underline{N}_i(t) = (N_{i1}(t), N_{i2}(t), \ldots, N_{im}(t))$$

denote the m-vector with entries

(1.17) $N_{ij}(t)$ = total progeny of type j born by t in the above critical m-type process starting with one new cell of type i.

Denote

$$H_i(\underline{s},t) \equiv E\left[s_1^{N_{i1}(t)} s_2^{N_{i2}(t)} \cdots s_m^{N_{im}(t)}\right]. \tag{1.18}$$

For $\underline{k} = (k_1,\ldots,k_m)$ a vector of non-negative integers, it is to be shown that as $t \to \infty$

$$P[\underline{Z}_i(t) = \underline{k}] \sim \frac{c}{t^2} \tag{1.19}$$

for $\underline{k} \neq 0$, and

$$P[\underline{N}_i(t) = \underline{k}] \sim c > 0, \tag{1.20}$$

where the constants c may be found by a recursive method.

II. Integral Equations and Approximations

Denote

$$P_i(t) = P[\underline{Z}_i(t) > \underline{0}] = \left(\sum_{i=1}^{m} u_i v_i \mu_i\right) u_i / Q(\underline{u}), \tag{2.1}$$

where $\underline{0} = (0,\ldots,0)$ and $\underline{u} > \underline{v}$ means term by sterm strict inequality for vectors, and

$$Q(\underline{u}) = \frac{1}{2} \sum_{i=1}^{m} \sum_{r=1}^{m} \sum_{\ell=1}^{m} h_{i\ell r}(1) u_\ell u_r v_i < \infty. \tag{2.2}$$

See [3].

Denote by $\underline{1 - P(t)}$ the m-vector

(2.3) $$\underline{1 - P(t)} = (1 - P_1(t), 1 - P_2(t), \ldots, 1 - P_m(t)).$$

Theorem 1. Under the assumptions of section I, for $\underline{k} = (k_1, \ldots, k_m)$ an m-vector of non-negative integers not all of whose terms are zero, for $t \to \infty$

(2.4) $$P[\underline{Z}_i(t) = \underline{k}] \sim \frac{c}{t^2}$$

where $c > 0$ is a constant depending on $\underline{k}$, i.

Proof. First consider the case $\underline{k} = \underline{e}_j$, that is, $\underline{k} = \underline{e}_j = (0,0,\ldots 0 1 \ldots 00)$ where the 1 is in the jth place, all other places have a zero.

From [3]

(2.5) $$F_i(\underline{s},t) = s_i(1-G_i(t)) + \int_0^t h_i(\underline{F}(\underline{s},t-u))dG_i(u),$$

where

(2.6) $$\underline{F}(\underline{s},t) = (F_1(\underline{s},t), F_2(\underline{s},t) \ldots F_m(\underline{s},t)).$$

Under the assumptions of the Theorem,

(2.7) $$P_{ij}(t) \equiv \left.\frac{\partial F_i(\underline{s},t)}{\partial s_j}\right|_{\underline{s}=\underline{0}} = P[\underline{Z}_i(t) = \underline{e}_j].$$

Applying (2.7) to (2.5) yields, for δ_{ij} the Kroneker delta,

(2.8) $$P_{ij}(t) = \delta_{ij}(1-G_i(t)) + \int_0^t \sum_{\ell=1}^m h_{i\ell}(\underline{F}(\underline{0},t-u))P_{\ell j}(t-u)dG_i(u),$$

all $1 \leq i, j \leq m$.

The proof for the $_{ij}(t)$ will continue by a series of claims.

Claim I. For arbitrary $c > 0$, $b > 0$, sufficiently large $T > 0$, the functions $Q_{ij}(t)$ defined for $1 \le i, j \le m$ by

$$(2.9) \qquad Q_{ij}(t) \equiv \begin{cases} cu_i t^{-(2-\epsilon)}, & t > T \\ b & , t < T \end{cases}$$

for some $0 < \epsilon < 1$, satisfy, for all $1 \le i, j \le m$, and all t sufficiently large,

$$(2.10) \qquad \sum_{i=1}^{m} v_i Q_{ij}(t) > v_j(1-G_j(t)) + \int_0^t \sum_{i=1}^{m} v_i \sum_{\ell=1}^{m} h_{i\ell}(\underline{F}(0),t-u))Q_{\ell j}(t-u)dG_i(u).$$

Proof of Claim I. Writing the integral term on the r.h.s. of (2.10) as $\int_0^{t/2} + \int_{t/2}^{t}$ for $t \gg 2T$, note that by (2.9), a Taylor expansion of $h_{i\ell}$ about $\underline{1}$, and a Newton expansion of $(t-u)^{-(2-\epsilon)}$ until the power $t^{-(3-\epsilon)}$ yields, as

$$(2.11) \qquad \underline{F}(\underline{0},t-u) \equiv \underline{1-P(t-u)},$$

$$(2.12) \qquad \int_0^{t/2} \sum_{i=1}^{m} v_i \sum_{\ell=1}^{m} \left[h_{i\ell}(\underline{1}) - \sum_{r=1}^{m} P_r(t-u)h_{i\ell r}(\underline{1}) + o(\sum_{r=1}^{m} P_r(t)) \right] \times \left[\frac{cu_\ell}{(t-u)^{2-\epsilon}} \right] dG_i(u) + o(t^{-4-\delta})$$

for some $\delta > 0$, and hence (2.10) equals, to terms of order $t^{-(3-\epsilon)}$,

$$(2.13)\qquad \int_0^{t/2} \sum_{i=1}^m v_i \left[\frac{cu_i}{t^{2-\epsilon}} \left(1 + \frac{(2-\epsilon)u}{t} + o\left(\frac{1}{t}\right)\right) \right] dG_i(u)$$

$$- \int_0^{t/2} \frac{\left(\sum_{i=1}^m \sum_{r=1}^m \sum_{\ell=1}^m v_i u_\ell u_r h_{i\ell r}(\underline{1})\right)\left(\sum_{i=1}^m u_i v_i \mu_i\right)}{t^{(3-\epsilon)} Q(\underline{u})} dG_i(u) + o(t^{-(3-\epsilon)})$$

using (1.9), (2.1), (2.2).

$$(2.14)\qquad 1-G_r(t) = o(t^{-(4-\delta)}) \quad \text{for } 1 \leq r \leq m$$

$$(2.15)\qquad \int_0^{t/2} u dG_i(u) = \mu_i + o(t^{-(3+\delta)}) \quad \text{for } 1 \leq i \leq m,$$

use of (2.14), (2.15) in (2.12), (2.13) yields the result of the Claim I.

<u>Claim II</u>. If for $1 \leq i, j \leq m$, $R_{ij}(t)$ are differentiable and

$$(2.16)\qquad R_{ij}(0) = 0$$

$$(2.17)\qquad \sum_{i=1}^m v_i R_{\ell j}(t) \underset{(<)}{>} \int_0^t \sum_{i=1}^m v_i \sum_{\ell=1}^m h_{i\ell}(\underline{1-P(t-u)}) R_{\ell j}(t-u) dG_i(u)$$

for all t sufficiently large, then

$$(2.18)\qquad \sum_{i=1}^m v_i R_{ij}(t) \underset{(<)}{>} 0$$

for all t sufficiently large.

<u>Proof of Claim II</u>. Expand $h_{i\ell}(\underline{1-P(t-u)})$ in a Taylor expansion about $h_{i\ell}(\underline{1})$. The r.h.s. of (2.17) becomes, after splitting the integral into two parts $\int_0^{t/2} + \int_{t/2}^t$,

$$(2.19)\qquad \int_0^{t/2} \sum_{i=1}^m v_i \left[h_{i\ell}(\underline{1}) - \sum_{r=1}^m P_r(t-u) h_{i\ell r}(\underline{1}) + o\left(\sum_{r=1}^m P_r(t/2)\right)\right] R_{\ell j}(t-u)\,dG_i(u)$$

$$+ o(t^{-(4+\delta)}),$$

which simplifies by (1.9), and plugging back into (2.17), where we now let

$$(2.20)\qquad R_j(t) \equiv \sum_{i=1}^m v_i R_{ij}(t),$$

that, for all t sufficiently large, assuming (say) the upper inequality in (2.17),

$$(2.21)\qquad R_j(t) > \int_0^t R_j(t-u)\,dG_i(u).$$

and

$$(2.22)\qquad R_j(0) = 0.$$

First note that from (2.21), it cannot be the case that $R_j(t) \equiv 0$ on any interval $[0,T]$. For if so, set $t = T$ in (2.21) to obtain $0 > 0$, a contradiction.

Suppose now that there is a first $t_o > 0$ such that $R_j(t_o) = 0$, $R_j(t) \neq 0$, $t < t_o$. Set $t = t_o$ in (2.21) to obtain a contradiction if, in fact, $R_j(t) > 0$ for $t < t_o$. If $R_j(t) < 0$ for $t < t_o$, by continuity there is a $T \leq t_o$ such that $|R_j(T)| > |R_j(t)|$ all $t < t_o$. Set $t = T$ in (2.21) to obtain an immediate contradiction.

This contradiction yields that $R_j(t) \neq 0$, $t > 0$. Hence if the upper inequality of (2.18) fails, it must be that $R_j(t) < 0$, all $t > 0$. Again, for any interval $[0,t_1]$, there is a $t_2 \leq t_1$ such that $|R(t_2)| > |R(t)|$, $t \leq t_1$.

Set $t = t_2$ in (2.21) to obtain a contradiction. This suffices to prove Claim II, as the argument for the lower inequality is similar.

Claim III. The functions defined for arbitrary $a \geq 0$, $b > 0$, $T > 0$,

$$R_{ij}(t) \equiv \begin{cases} a \quad , & t < T \\ \dfrac{bu_i}{t^2}, & t > T \end{cases} \tag{2.23}$$

satisfy, for $1 \leq i, j \leq m$, the system

$$R_{ij}(t) = f_{ij}(t) + \int_0^t \sum_{\ell=1}^m h_{i\ell}(\underline{1-P(t-u)})R_{\ell j}(t-u)dG_i(u) \tag{2.24}$$

where, as $t \to \infty$,

$$f_{ij}(t) = o(t^{-3}). \tag{2.25}$$

Proof of Claim III. The integral on the r.h.s. of (2.24) with (2.23) substituted becomes, after a Taylor expansion about $h_{i\ell}(\underline{1})$, less than or equal to

$$\int_0^{t/2} \sum_{\ell=1}^m (h_{i\ell}(\underline{1}) - \sum_{r=1}^m P_r(t-u)h_{i\ell r}(\underline{1}) + o(\sum_{r=1}^m P_r(t/2)))\frac{bu_\ell}{(t-u)^2}\, dG_i(u) \tag{2.26}$$

$$+ G(t) - G(t/2)$$

and similarly, the integral on the r.h.s. of (2.24) is greater than or equal to the integral term of (2.26).

The integral in (2.26) equals

$$(2.27)\quad b\int_0^{t/2}\left[\frac{u_i}{t^2}\left(1+\frac{2u}{t}+o\left(\frac{1}{t}\right)\right) - \sum_{\ell=1}^{m}\sum_{r=1}^{m}\frac{h_{i\ell r}(\underline{1})\left(\sum_{\alpha=1}^{m}\mu_\alpha u_\alpha v_\alpha\right)u_r u_\ell}{Q(\underline{u})t^3}\right] \times dG_i(u)$$

$$+ o(t^{-3}).$$

Since

$$(2.28)\qquad \int_0^{t/2} u\,dG_i(u) = \mu_i + o(t^{-3}),$$

and

$$(2.29)\qquad G(t) - G(t/2) = o(t^{-4}),$$

this suffices for the proof of Claim III.

The proof of Theorem I may now be completed.

Define the iterative sequence for n = 0,1,2,...

$$(2.30)\quad P_{(n+1)ij}(t) \equiv \delta_{ij}(1-G_i(t)) + \int_0^t \sum_{\ell=1}^{m} h_{i\ell}(\underline{1-P(t-u)})P_{(n)\ell j}(t-u)dG_i(u)$$

with

$$(2.31)\quad P_{(0)}(t) \equiv P_{(0)ij}(t) \equiv R_{ij}(t) = \frac{bu_i}{t^2},\quad \text{for t sufficiently large.}$$

Then, suppressing the i,j part of the subscript,

$$(2.32)\quad |P_{(1)}(t) - P_{(0)}(t)| = |\delta_{ij}(1-G_i(t)) - f_{ij}(t)| = o(t^{-3}),$$

which implies that

(2.33) $$P_{(1)ij}(t) \sim \frac{cu_i}{t^2} \quad \text{for some } c > 0.$$

Assume, by an induction hypothesis, that, for some constant c (the value of c may change from expression to expression)

(2.34) $$P_{(\ell)ij}(t) \sim \frac{cu_i}{t^2} \quad \text{for } 0 \leq \ell \leq n-1.$$

Then, by (1.9),

(2.35) $$|P_{(n)ij}(t)-P_{(n-1)ij}(t)| \leq \left|\int_0^{t/2} \sum_{\ell=1}^{m} h_{i\ell}(\underline{1-P(t-u)})P_{(n-1)\ell j}(t-u)dG_i(u)-P_{(n-1)ij}(t)\right|$$

$$+ 2b(1-G(t/2)).$$

Again by (1.9) and a Taylor expansion about $h_{i\ell}(\underline{1})$,

(2.36) $$|P_{(n)ij}(t) - P_{(n-1)ij}(t)| \leq \left|\int_0^{t/2} \sum_{\ell=1}^{m} \left[h_{i\ell}(\underline{1}) - \sum_{r=1}^{m} P_r(t-u)h_{i\ell r}(\underline{1})\right.\right.$$

$$\left.\left. + o(\sum_{r=1}^{m} P_r(t/2))\right] P_{(n-1)\ell j}(t-u)dG_i(u) - P_{(n-1)ij}(t)\right| +2b(1-G(t/2))$$

or

(2.37) $$|P_{(n)ij}(t) - P_{(n-1)ij}(t)| \leq \left|c\int_0^{t/2} \sum_{\ell=1}^{m} \left[h_{i\ell}(\underline{1}) - \sum_{r=1}^{m} P_r(t-u)h_{i\ell r}(\underline{1})\right.\right.$$

$$\left.\left. + o(\sum_{r=1}^{m} P_r(t/2))\right] \frac{u_\ell}{(t-u)^2}\, dG_i(u) - \frac{cu_i}{t^2}\right| + 2b(1-G(t/2)),$$

where the two constants labeled c in (2.37) have the same value.

Hence

(2.38) $$|P_{(n)ij}(t) - P_{(n-1)ij}(t)| \le \left| c\int_0^{t/2} \left[\frac{u_i}{t^2}(1 + \frac{2u}{t} + o(\frac{1}{t})) + \left|0(\frac{1}{t^3})\right| \right] dG_i(u) - \frac{cu_i}{t^2}\right| + o(t^{-3})$$

or

(2.39) $$|P_{(n)ij}(t) - P_{(n-1)ij}(t)| \le 0(\frac{1}{t^3}),$$

which implies that, for $t \to \infty$

(2.40) $$P_{(n)ij}(t) \sim \frac{cu_i}{t^2},$$

and

(2.41) $$P_{(n)ij}(t) \le \frac{cu_i}{t^2}$$ for appropriate choice of c.

Claims I, II yield that for arbitrary $c > 0$, one may write

(2.42) $$P_{ij}(t) < \frac{cu_i}{t^{2-\epsilon}}$$

for all t.

Hence

(2.43) $$|P_{ij}(t) - P_{(n)ij}(t)| \le \int_0^t \sum_{\ell=1}^m h_{i\ell}\underline{(1-P(t-u))}\,|P_{\ell j}(t-u) - P_{(n-1)\ell j}(t-u)|\,dG_i(u).$$

Denote

$$\Delta_{(n)}(t) \equiv |P_{ij}(t) - P_{(n)ij}(t)|. \tag{2.44}$$

Then, using (2.40),(2.41),(2.42) in (2.43) yields

$$\Delta_{(n)}(t) \le \int_0^t \Delta_{(n-1)}(t-u)dG_i(u) \tag{2.45}$$

Iterating (2.45), and denoting

$$\Delta * G \equiv \int_0^t \Delta(t-u)dG(u), \tag{2.46}$$

$$\Delta_{(n)}(t) \le \Delta_0 * G_i^{(n)}(t) \le G_i^{(n)}(t) \tag{2.47}$$

where

(2.48) $G_i^{(n)}(t)$ is the n-th convolution of $G_i(t)$.

Let $\{X_\ell\}$ be a sequence of I.I.D. r.v.s. each with distribution function $G_i(t)$, and let

$$S_n \equiv \sum_{i=1}^{n} X_i . \tag{2.49}$$

Then

$$G_i^{(n)}(t) \equiv P[S_n \le t] = P[S_n - n\mu_i < t - n\mu_i]. \tag{2.50}$$

By Chebyshev's inequality,

$$(2.51)\qquad G_i^{(n)}(t) \le \frac{\operatorname{Var} S_n}{(n\mu_i - t)^2} \le \frac{Kn}{(n\mu_i - t)^2}\,.$$

Let $n \ge t^{2+\varepsilon}$. Then (2.47) yields, for each t,

$$(2.52)\qquad \Delta_{(n)}(t) \le t^{-(2+\varepsilon)}\,.$$

Then (2.52),(2.40), yield that

$$(2.53)\qquad P_{ij}(t) \sim \frac{c}{t^2}$$

for some $c > 0$.

Hence the theorem is proved for $\underline{k} = \underline{e}_j$.

In general, note that for $\underline{k} = (k_1,\ldots,k_m)$ and $\sum_{i=1}^{m} k_i = k$, where k is the degree of differentiation,

$$(2.54)\quad P_{\underline{k}}(t) \equiv \frac{1}{k_1!k_2!\cdots k_m!}\ \frac{\partial^k}{\partial s_1^{k_1}\cdots \partial s_m^{k_m}} F_i(\underline{s},t)\Big|_{\underline{s}=\underline{0}} = P[\underline{Z}_i(t) = \underline{k}]\,.$$

Then, successively differentiating (2.5) according to (2.54) will yield, by Leibniz' rule for successive differentiation

$$(2.55)\qquad P_{i\underline{k}}(t) = f_{i\underline{k}}(t) + \int_0^t \sum_{\ell=1}^{m} h_{i\ell}\underline{(1-P(t-u))}P_{\ell\underline{k}}(t-u)\,dG_i(u),$$

where $f_{i\underline{k}}(t)$ consists of sums of terms of the form

$$(2.56)\quad \int_0^t \sum_{\ell,r,\cdots=1}^{m} \left(\frac{\partial^p}{\partial s_\ell \partial s_r \cdots} h_i(\underline{F}(\underline{s},t-u))\Big|_{\underline{s}=0} \right) P_{\ell \underline{q}}(t-u) P_{r\underline{b}}(t-u) \cdots dG_i(u)$$

where

(2.57) $p < k$ and the sum of degrees of $\underline{q}$, $\underline{b}$,$\cdots$ total less than k.

By induction, assume that for degree of $\underline{k} \leq n$

$$(2.58)\quad P_{i\underline{k}}(t) \sim \frac{c}{t^2} \quad \text{for } 1 \leq i \leq m$$

where $c > 0$ may change in value.

By (2.55)-(2.58), (2.55) may be written

$$(2.59)\quad P_{i\underline{k}}(t) = f_{i\underline{k}}(t) + \int_0^t \sum_{\ell=1}^{m} h_{i\ell}\underline{(1-P(t-u))} P_{\ell\underline{k}}(t-u)dG_i(u)$$

where

$$(2.60)\quad f_{i\underline{k}}(t) = o(t^{-3}).$$

Hence the techniques of section 2 used in the proof of the theorem for $P_{ij}(t)$ may be used again, since (2.59),(2.60) is of the form (2.8).

Hence Theorem 1 is proved.

Theorem 2. Under the assumptions of section I, for $\underline{k} = (k_1,\ldots,k_m)$ a vector of strictly positive integers, as $t \to \infty$,

$$(2.61)\quad P[\underline{N}_i(t) = \underline{k}] \sim c \geq 0$$

where the constant c will depend on $\underline{k}$,i and can be obtained by a recursive argument.

Proof. Denote for $1 \leq i \leq m$,

$$H_i(\underline{s},t) \equiv H_i(s_1,\ldots,s_m,t) \equiv E(s_1^{N_{i1}(t)} \cdots s_m^{N_{im}(t)}). \tag{2.62}$$

The law of total probability yields

$$H_i(\underline{s},t) = s_i(1-G_i(t) + \int_0^t h_i(\underline{H}(\underline{s},t-u))dG_i(u)), \tag{2.63}$$

where

$$\underline{H}(\underline{s},t) = (H_1(\underline{s},t),H_2(\underline{s},t),\ldots,H_m(\underline{s},t)). \tag{2.64}$$

Since

$$Q_{ij}(t) \equiv \left.\frac{\partial H_i(\underline{s},t)}{\partial s_j}\right|_{\underline{s}=\underline{0}} = P[\underline{N}_i(t) = \underline{e}_j], \tag{2.65}$$

and note that

$$H_i(\underline{0},t) = 0, \tag{2.66}$$

it follows immediately that

$$Q_{ij}(t) \equiv \delta_{ij}(1-G_i(t) + h_i(0)G_i(t)) \sim \delta_{ij}h_i(0) \tag{2.67}$$

as $t \to \infty$.

By an induction, assume, for vector $\underline{k} = (k_1,\ldots,k_m)$ of degree $k = \sum_{i=1}^{m} k_i$, for $k \leq n$

$$(2.68) \qquad Q_{i\underline{k}}(t) \equiv P[\underline{N}_i(t) = k] \sim c.$$

Then for degree $r = n+1$, note that, by Leibniz' rule, $Q_{i\underline{r}}(t)$ is the sum of terms of the form

$$(2.69) \quad \int_0^t \sum_{\ell,r,q,\cdots=1}^{m} \left.\frac{\partial^k h_i(\underline{H}(\underline{s},t-u))}{\partial s_\ell \partial s_r \partial s_q \cdots}\right|_{\underline{s}=\underline{0}} Q_{\ell\underline{a}}(t-u) Q_{r\underline{\eta}}(t-u) Q_{q\underline{\beta}}(t-u)\, dG_i(u)$$

possibly premultiplied by $a\delta_{i\eta}$, the Kronecker delta, each term Q with degree $k \leq n$, (in fact the sum of all Q-degrees is $\leq n$) the induction hypothesis applies to the Q's in the integrand of each term (2.69), as $\underline{H}(\underline{0},t) = \underline{0}$, it follows that

$$(2.70) \qquad \left.\frac{\partial^k h_i(\underline{H}(\underline{s},t-u))}{\partial s_\ell \partial s_r \partial s_q \cdots}\right|_{\underline{s}=0} = h_{i\ell rq\ldots}(\underline{0}), \quad \text{a constant.}$$

This discussion and (2.69),(2.70) suffices to prove Theorem 2.

III. Remarks and Extensions.

The analog of Claim III in one dimension could have been used to simplify the argument of [4], and in particular, to not require the "$\log t/t^2$" condition of [1].

The argument using Claims I, II to deduce that $P_{ij}(t) < \frac{u_i}{t^{2-\epsilon}}$ could have been omitted, and it would have sufficed to note that $P_{i\underline{k}}(t) \leq P_i(t) \equiv P[Z_i(t) > 0] \sim \frac{c}{t}$, but Claims I, II make the method self-contained and show how finer estimates can be easily obtained.

The arguments of this paper can be easily extended to obtain asymptotic marginal probabilities as follows.

Theorem III. *Under the assumptions of section I, for I a subset of the integers* $\{1,2,\ldots,m\}$, *as* $t \to \infty$

(3.1) $$P[Z_{ij}(t)=k_j,\ \text{all } j \in I] \sim P[\underline{Z}_i(t)=\underline{k}] \sim \frac{c}{t^2}$$

where $k_j > 0$ *for* $j \in I$,

(3.2) $$\underline{k} = (k_1\delta_{1I}, k_2\delta_{2I}, \ldots, k_m\delta_{mI})$$

and

(3.3) $$\delta_{iI} = \begin{cases} 1 & \text{if } i \in I \\ 0 & \text{if } i \notin I. \end{cases}$$

Also, as $t \to \infty$,

(3.4) $$P[N_{ij}(t)=k_j;\ \text{all } j \in I] \sim c > 0.$$

Outline of Proof. To prove (3.1), a modification of the Theorem 1 of [3] is required, as given by the following claim.

Claim. For I a proper subset of the integers $\{1,2,\ldots,m\}$, as $t \to \infty$,

(3.5) $$P_{iI}(t) \equiv P[Z_{ij}(t) > 0 \quad \text{for all } j \in I] \sim P[\underline{Z}_i(t) > \underline{0}].$$

Outline of Proof of Claim. Denote

$$(3.6)\qquad \begin{aligned} J &= \{1,2,\ldots,m\} \\ \underline{1}_I &= (\delta_{1I},\delta_{2I},\ldots,\delta_{mI}) \\ \underline{1}-\underline{1}_I = \underline{1}_{J-I} &= (1-\delta_{1I},1-\delta_{2I},\ldots,1-\delta_{mI}). \end{aligned}$$

Then (3.5) yields, for $\underline{s} = \underline{1}_{J-I}$

$$(3.7)\quad 1-P_i(t) = \delta_{i(J-I)}(1-G_i(t)) + \int_0^t h_i(1-P_{1I}(t-u),\ldots,1-P_{m,I}(t-u))dG_i(u).$$

The argument for $P_i(t)$ in [3] pp. 527-529, may be modified for $P_{iI}(t)$ by noting that the basic idea of bounding $P_{iI}(t)$ above and below by probabilities of non-extinction for appropriate one-dimensional critical age-dependent processes, with known behavior, may be retained. A sketch of the modifications required is as follows. Let

$$(3.8)\qquad \begin{aligned} G_\alpha(t) &= \max_{1\le i\le m} G_i(t) \\ G_\beta(t) &= \min_{1\le i\le m} G_i(t). \end{aligned}$$

The critical m-type processes $\overline{Z}_i(t)$, $\overline{\overline{Z}}_i(t)$ defined on p. 528 of [3] remain the same, both with generating functions $h_i(\underline{s})$, $1 \le i \le m$ and with progeny lifetime distributions $G_\alpha(t)$, $G_\beta(t)$ respectively.

Then denoting by $\overline{P}_{iI}(t)$, $\overline{\overline{P}}_{iI}(t)$ the quantities corresponding to $P_{iI}(t)$ for the $\overline{Z}_i(t)$, $\overline{\overline{Z}}_i(t)$ processes respectively, it follows as before that

$$\overline{P}_{iI}(t) \le P_{iI}(t) \le \overline{\overline{P}}_{iI}(t) \tag{3.9}$$

and also (3.7) holds for $\overline{P}_{iI}(t)$, $\overline{\overline{P}}_{iI}(t)$ with $G_i(t)$ replaced by $G_\alpha(t)$, $G_\beta(t)$ for all $1 \le i \le m$, respectively.

In [3] p. 528, referring to the numbering system there, inequalities (3.14), (3.15) hold as before, and relations (3.10), (3.12), hold with the term

$$\sum_{i \in J-I} \frac{v_i}{a} (1-G_\alpha(t) \equiv c(1-G_\alpha(t)) \tag{3.10}$$

added to the respective r.h.s., and subtracted from the r.h.s. of (3.16), and relations (3.11), (3.13) hold with

$$\sum_{i \in J-I} \frac{v_i}{a} (1-G_\beta(t) \equiv c(1-G_\beta(t)) \tag{3.11}$$

added to the respective r.h.s., and subtracted from the r.h.s, of (3.17),

$$0 < c < 1. \tag{3.12}$$

The basic results of Lemmas 2, 7, 8 of [1] used in [3] p. 530 remain true if the term 1-G(t) is replaced by (1-c)(1-G(t)) in equation (2.3) p. 385, (3.2) p. 389, (3.3) p. 389 of [1]. That lemma 2 of [1] still holds is clear since the argument there depends solely on monotonicity considerations unaffected by this substitution. The new lemma 7 of [1] p. 389 should now be stated as:

If

$$x(t) - a = (1-c)(1-G(t)) + \int_0^t x(t-y)dG(y), \tag{3.13}$$

Then

$$(3.14) \qquad \frac{x(t)}{t} = \frac{a}{\mu} + 0(\frac{1}{t})$$

where

$$(3.15) \qquad \mu = \int_0^\infty t dG(t),$$

and as $t \to \infty$

$$(3.16) \qquad t^3(1-G(t)) \to 0.$$

That this follows is well-known or can be seen by the methods of this paper as follows.

One may verify that

$$(3.17) \qquad f(t) \equiv \frac{a}{\mu} t$$

satisfies

$$(3.18) \qquad f(t) = a + r(t) + \int_0^t f(t-u)dG(u)$$

where

$$(3.19) \qquad r(t) = o(\frac{1}{t^2}).$$

Then, using the iteration scheme

$$(3.20) \qquad x_{(n)}(t) = (1-c)(1-G(t)) + \int_0^t x_{(n-1)}(t-u)dG(u)$$

it may be verified that as $t \to \infty$, for

$$x_{(0)}(t) \equiv at/\mu \,, \tag{3.21}$$

$$\frac{1}{t}\left|x_{(n)}(t) - x_{(n-1)}(t)\right| = o(1) \tag{3.22}$$

for all n, and

$$\frac{1}{t}\left|x(t) - x_{(n)}(t)\right| = o(1) \tag{3.23}$$

for all n sufficiently large with respect to t.

Then (3.17)-(3.23) yield the new lemma 7.

The lemma 8 uses lemma 7, so remains true since the new lemma 7 has the same conclusion.

The other results embodied in revised lemmas of [1] remain true for reasons of monotonicity or convexity which are unchanged, or by reasoning as just above.

Hence the results of [3] pp. 530 ff. remain true with the indicated modifications and (3.5) holds. But then (2.8) holds with the argument

$$h_{i\ell}(\underline{1-P}_{J-I}(t)) \sim h_{i\ell}(\underline{1-P(t)}) \tag{3.24}$$

where

$$\underline{1-P}_{J-I}(t) = (1-P_{1,J-I}(t), 1-P_{2,J-I}(t), \ldots, 1-P_{m,J-I}(t)), \tag{3.25}$$

and all the results of Theorem 1 go through as before with modified (2.8) written

$$P_{ijI}(t) = \delta_{ij}(1-G_i(t)) + \int_0^t \Sigma\, h_{i\ell}(\underline{1-P_{J-I}(t-u)})P_{\ell jI}(t-u)dG_i(u). \tag{3.26}$$

This suffices to indicate the essential modifications that yield (3.1). To indicate (3.4), an example will illustrate the general method. Let

(3.27) $$H_{ij}(t) \equiv P[N_{ij}(t) = 1].$$

Note that

(3.28) $$\left.\frac{\partial H_i(\underline{s},t)}{\partial s_j}\right|_{\underline{s}=\underline{1-e}_j} = H_{ij}(t)$$

where

(3.29) $\underline{1-e}_j = (1110111)$ with the zero in the jth place.

Apply (3.28) to (2.63) to yield, noting

(3.30) $$H(\underline{1-e}_j, t) = 0,$$

that

(3.31) $$H_{ij}(t) = \delta_{ij}h_i(0)G_i(t) + (1-\delta_{ij})\int_0^t \sum_{\ell=1}^{m} h_{i\ell}(0)H_{\ell j}(t-u)dG_i(u).$$

A consistent sequenct $\{c_{ij}\}$ satisfying, for $1 \leq i, j \leq m$

(3.32) $$c_{ij} = \delta_{ij}h_i(0) + (1-\delta_{ij}) \sum_{\ell=1}^{m} h_{i\ell}(0)c_{\ell j}$$

is constructed, since note that

(3.33) $$1 \geq H_{ij}(t) \geq Q_{ij}(t) = P[\underline{N}_i(t) = \underline{e}_j] \to c > 0,$$

Then the method of Theorem 1 may be employed by defining the iterative scheme

$$(3.34)\quad H_{ij(n+1)}(t) = \delta_{ij}h_i(0)G_i(t) + (1-\delta_{ij})\int_0^t \sum_{\ell=1}^{m} h_{i\ell}(0)H_{\ell j(n)}(t-u)dG_i(u)$$

with

$$(3.35)\qquad H_{ij(0)}(t) \equiv c_{ij}.$$

It remains to show that, for $t \to \infty$

$$(3.36)\qquad \left|H_{ij(n+1)}(t) - H_{ij(n)}(t)\right| = o(1)$$

for all n, and

$$(3.37)\qquad \left|H_{ij}(t) - H_{ij(n)}(t)\right| = o(1)$$

for sufficiently large n with respect to t. Thus (3.28)-(3.37) suffice for (3.27). The general case follows by a tedious induction on orders of derivatives, using Leibniz' rule to deduce the general form of the integral equation for the quantity (3.4) obtained by taking appropriate partial derivatives of (2.63), following the arguments from (2.65) to (2.69), and will not be given.

The method can be applied to multi-type sub- and supercritical cases also.

REFERENCES

[1] CHOVER, J. and NEY, P. (1968). The non-linear renewal equation. J. Analyse Math. **21**, 381-413.

[2] KARLIN, S. (1966). *A First Course in Stochastic Processes*. Academic Press, New York.

[3] WEINER, H.J. (1970). On a multi-type critical age-dependent branching process. *J. Applied Prob.*, **7**, 523-543.

[4] WEINER, H.J. (1976). Asymptotic probabilities in a critical age-dependent branching process. *J. Applied Prob.* (to appear).

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (*When Data Entered*)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER #2	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (*and Subtitle*) LIMIT PROBABILITIES IN A MULTI-TYPE CRITICAL AGE-DEPENDENT BRANCHING PROCESS		5. TYPE OF REPORT & PERIOD COVERED Technical Report 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Howard J. Weiner		8. CONTRACT OR GRANT NUMBER(s) DAAG29-77-G-0031
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Statistics Stanford University Stanford, California 94305		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS P-14435-M
11. CONTROLLING OFFICE NAME AND ADDRESS U.S. Army Research Office Post Office Box 12211 Research Triangle Park, NC 97709		12. REPORT DATE March 7, 1977 13. NUMBER OF PAGES 25
14. MONITORING AGENCY NAME & ADDRESS(*if different from Controlling Office*)		15. SECURITY CLASS. (*of this report*) Unclassified 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (*of this Report*)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (*of the abstract entered in Block 20, if different from Report*)

18. SUPPLEMENTARY NOTES

The findings in this report are not to be construed as an official Department of the Army position, unless so designated by other authorized documents. This report partially supported under Office of Naval Research Contract N00014-76-C-0475 (NR-042-267) and issued as Technical Report No. 242

19. KEY WORDS (*Continue on reverse side if necessary and identify by block number*)

LIMIT - PROBABILITY - MULTI-TYPE - CRITICAL - BRANCHING PROCESSES

20. ABSTRACT (*Continue on reverse side if necessary and identify by block number*)

(see reverse side)

Limit Probabilities in a Multi-type Critical Age-Dependent Branching Process

by

Howard J. Weiner

ABSTRACT

Let $\underline{Z}_i(t) \equiv (Z_{i1}(t), Z_{i2}(t), \ldots, Z_{im}(t))$ where $Z_{ij}(t)$ = number of cells of type j at t starting at $t = 0$ with one new cell of type i, with $1 \leq i, j \leq m$. Assuming this to be an m-type critical age-dependent branching process, for $\underline{k} \equiv (k_1, \ldots, k_m)$ an m-vector of non-negative integers, not all of which are zero, it is shown that as $t \to \infty$, $P[\underline{Z}_i(t) = \underline{k}] \sim \frac{c}{t^2}$ for some $c > 0$.

Similarly, let $\underline{N}_i(t) = (N_{i1}(t), N_{i2}(t), \ldots, N_{im}(t))$ denote the m-vector with entries $N_{ij}(t)$ = total progeny of type j born by t in the critical m-type process starting with a new cell of type i at $t = 0$.

For $\underline{k} = (k_1, \ldots, k_m)$ an m-vector of non-negative integers, it is shown that as $t \to \infty$, $P[\underline{N}_i(t) = \underline{k}] \sim c > 0$ where the constants may be evaluated explicitly by a recursion. Related results on marginal probabilities are indicated.

#2/#242